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1978 J. Phys. A: Math. Gen. 11 1963

(http://iopscience.iop.org/0305-4470/11/10/014)

J. Phys. A: Math. Gen., Vol. 11, No. 10, 1978. Printed in Great Britain

Massless scalar field in a de Sitter universe and its thermal flux

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Received 30 December 1977, in final form 2 May 1978

Abstract. The time dependent wave equation for a conformally coupled massless scalar field in a de Sitter universe is examined. It is pointed out that the exact solutions of the radial equation valid over the 'whole range' can be found—unlike the corresponding situation in the Schwarzschild spacetime. The power spectrum of Hawking radiation is evaluated after calculating the absorption probability at the horizon. The solution is seen to be well behaved at the origin.

1. Introduction

The possibility of thermal emission from black holes due to quantum effects was pointed out by Hawking (1974) and has attracted widespread attention. The thermal emission is linked to the existence of an event horizon. The possibility of such thermal emission in other systems with event horizons immediately suggests itself. This is indeed seen to be the case for the de Sitter universe with cosmological event horizon which has thermal radiation associated with itself (Gibbons and Hawking 1977). One is thus motivated to study various fields in de Sitter universe.

In this paper we give expressions for the thermal power emitted by the de Sitter universe with conformally coupled massless scalar field. The thermal character of the radiation and the equivalent temperature have been given by Gibbons and Hawking (1977) using quantum Green functions. Our problem reduces essentially to the calculation of the probability of absorption by the horizon of waves incident upon it. This is done following the procedure of Page (1976) and Teukolsky (1973) and involves essentially solving the classical wave equation in the de Sitter metric.

We find in § 2 exact solutions to the wave equation that are valid throughout the region $0 \leq r \leq a$ where a is the radius of the de Sitter horizon. This is in contrast to the corresponding situation in the Kerr–Newman background or even the Schwarzchild background where no exact solution is known to be valid over the whole range of radial coordinates (Rowan and Stephenson 1977).

In § 3 we briefly review Kruskal coordinates and the arguments leading to a temperature $k_B T = 1/(2\pi a)$ for the horizon. The energy spectrum is given in § 4 after calculating the absorption probability. This is done by picking the solution of the scalar wave equation satisfying the outgoing boundary condition at the horizon and identifying its ingoing and outgoing parts away from the horizon ($0 < r < a$). The ratio of the ingoing to the outgoing part gives us the reflection amptitude which in turn enables us to calculate the absorption probability.

0305-4770/78/0010-1963$01.00

In § 5 we satisfy ourselves that the behaviour of the amplitudes at the origin ($r=0$) is regular and also notice some differences from the black hole case.

2. The radial equation and its solution

We start with the equation

$$\Box\Phi=\tfrac{1}{6}R\Phi. \tag{2.1}$$

The metric being considered is written in the form

$$ds^2=-(1-r^2/a^2)\,dt^2+(1-r^2/a^2)^{-1}\,dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2) \tag{2.2}$$

where the radius of the de Sitter horizon, a, is related to the cosmological constant Λ and the curvature scalar, R, by

$$R=4\Lambda=12/a^2. \tag{2.3}$$

Using

$$\Box\Phi=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right)$$

we obtain

$$-(1-r^2/a^2)^{-1}\Phi_{,tt}+\frac{1}{r^2}\{r^2(1-r^2/a^2)\Phi_{,r}\}_{,r}+\frac{L^2}{r^2}\Phi=\frac{2}{a^2}\Phi \tag{2.4}$$

where L^2 is the flat-space angular-momentum operator. This equation can be separated by writing

$$\Phi=\frac{f_l(r)}{r}Y_{lm}(\theta,\phi)\exp(-i\omega t). \tag{2.5}$$

The function $f_l(r)$ can be easily seen to satisfy the radial equation (with $z\equiv a/r$)

$$(1-z^2)\frac{d}{dz}\left((1-z^2)\frac{df_l}{dz}\right)+[l(l+1)(1-z^2)-(ia\omega)^2]f_l=0 \tag{2.6}$$

This equation is just the associated Legendre equation that has known solutions valid over the range in which we are interested, i.e. $|z|>1$. In terms of the standard hypergeometric functions (Abramowitz and Stegun 1970) the solutions are

$$f_l^{(1)}=e^{-\pi a\omega}2^{-l-1}\pi^{-1/2}\Gamma(l+1+ia\omega)(r/a)^{l+1}(1-r^2/a^2)^{ia\omega/2}$$
$$\times{}_2F_1\left(1+\frac{l}{2}+\frac{ia\omega}{2},\frac{1}{2}+\frac{l}{2}+\frac{ia\omega}{2};l+\frac{3}{2};\frac{r^2}{a^2}\right) \tag{2.7a}$$

and

$$f_l^{(2)}=\frac{\Gamma(-\frac{1}{2}-l)(r/a)^{l+1}}{2^{l+1}\pi^{1/2}(1-r^2/a^2)^{ia\omega/2}\Gamma(-l-ia\omega)}{}_2F_1\left(\frac{1}{2}+\frac{l}{2}-\frac{ia\omega}{2},1+\frac{l}{2}-\frac{ia\omega}{2};l+\frac{3}{2};\frac{r^2}{a^2}\right)$$
$$+\frac{2^l\pi^{-1/2}\Gamma(\frac{1}{2}+l)(r/a)^{-l}}{(1-r^2/a^2)^{ia\omega/2}\Gamma(1+l-ia\omega)}{}_2F_1\left(-\frac{l}{2}-\frac{ia\omega}{2},\frac{1}{2}-\frac{l}{2}-\frac{ia\omega}{2};\frac{1}{2}-l;\frac{r^2}{a^2}\right). \tag{2.7b}$$

Defining a 'tortoise' coordinate r^*, by

$$\mathrm{d}r^* \equiv \frac{\mathrm{d}r}{(1-r^2/a^2)}, \qquad r^* = \frac{a}{2}\ln\left(\frac{a+r}{a-r}\right) \tag{2.8}$$

the solutions for equation (2.6) are expected to behave as $\exp(\pm \mathrm{i}\omega r^*)$ near $r = a$. That this is indeed the case can easily be seen from equation (2.7) which shows that $f_l^{(1)}$ and $f_l^{(2)}$ approach $\exp(-\mathrm{i}\omega r^*)$ and $\exp(+\mathrm{i}\omega r^*)$ respectively. Thus $f_l^{(1)}$ and $f_l^{(2)}$ represent incoming and outgoing solutions respectively at the de Sitter horizon.

3. Kruskal coordinates and Hawking temperature

For discussing the boundary conditions on the horizon, it is convenient to define (Gibbons and Hawking 1977) Kruskal type coordinates U and V with

$$r = a(1+UV)(1-UV)^{-1} \qquad \exp(2t/a) = -VU^{-1}. \tag{3.1}$$

In terms of $u = t - r^*$ and $v = t + r^*$ we have

$$V = \pm \mathrm{e}^{u/a} \qquad \text{and} \qquad U = \mp \mathrm{e}^{-v/a} \tag{3.2}$$

where upper and lower signs refer to first and second exterior regions respectively.

The Gibbons–Hawking vacuum is defined by functions which are positive frequency with respect to the Kruskal coordinate U but which are mixtures of positive and negative frequency with respect to coordinate t (or u and v). As $\exp(-\mathrm{i}\omega U)$ is analytic in the lower half U-plane it is readily seen (Unruh 1976) that the suitably normalised combination

$$\frac{\mathrm{e}^{\pi a\omega/2}|U^{I}|^{\mathrm{i}a\omega}}{[2\sinh(\pi a\omega)]^{1/2}} + \frac{\mathrm{e}^{-\pi a\omega/2}|U^{II}|^{\mathrm{i}a\omega}}{[2\sinh(\pi a\omega)]^{1/2}} \tag{3.3}$$

is also analytic in the lower half U-plane. Here U^{I} and U^{II} are non-vanishing for $U<0$ and $U>0$ respectively. $U<0$ refers to the normal or first exterior region while $U>0$ refers to the second exterior region and is negative frequency in coordinate t. On second quantising the field Φ we can expand it in the U-basis or t-basis. As is well known a vacuum with respect to the U-basis may contain particles with respect to the t-basis. The relation between the two bases gives us the relations between the corresponding creation and destruction operators.

Writing h_i, f_i and g_i for functions which are positive frequency with respect to U, t and V respectively we can expand Φ as (Hawking 1974)

$$\Phi = \sum_i (a_i f_i + a_i^+ \bar{f}_i + c_i g_i + c_i^+ \bar{g}_i) \tag{3.4}$$

or

$$\Phi = \sum_i (b_i h_i + b_i^+ \bar{h}_i). \tag{3.5}$$

If

$$h_i = \sum_j (\alpha_{ij} f_j - \beta_{ij} \bar{f}_j) \tag{3.6}$$

then

$$f_i = \sum_j (\bar{\alpha}_{ij} h_j + \bar{\beta}_{ij} \bar{h}_j) \tag{3.7}$$

and

$$a_i = \sum_i (\alpha_{ij} b_j + \beta_{ij} b_j^+). \tag{3.8}$$

If the detector is sensitive to f_i (positive frequency with respect to t) it measures $\langle a_i^+ a_i \rangle$. If the initial vacuum is defined by

$$b_i|0\rangle = 0. \tag{3.9}$$

We have using equation (3.8)

$$\langle 0|a_i^+ a_i|0\rangle = \sum_j |\beta_{ij}|^2. \tag{3.10}$$

From equation (3.3) we have

$$\sum_j |\beta_{ij}|^2 = \frac{e^{-\pi a \omega}}{2 \sinh(\pi a \omega)} = \frac{1}{e^{2\pi a \omega} - 1} \tag{3.11}$$

which corresponds to a Bose gas with temperature T given by

$$k_B T = 1/(\pi a). \tag{3.12}$$

On the right-hand side of equation (3.3) we have taken

$$\beta_{\omega\omega'} \propto f(\omega) \delta_{\omega\omega'} \qquad \text{(as also for } \alpha_{\omega\omega'}\text{).}$$

4. Boundary condition at the horizon and the energy spectrum

The power emitted is given by

$$\frac{dE}{dt\, d\omega} = \frac{1}{2\pi} \sum_l (2l+1) \frac{\Gamma_l(\omega)\omega}{e^{2\pi a \omega} - 1} \tag{4.1}$$

analogous to black-hole formula (Hawking 1974, Page 1976) where $\Gamma_l(\omega)$ is the probability of absorption of a wave incident on the horizon.

In calculating Γ_l the solution of interest at the horizon is the outgoing wave, i.e. $\exp(+i\omega r^*)$. This also follows from the fact that we have $\exp(-i\omega V)$ at the future horizon which is a function of $u = t - r^*$ only. This boundary condition is satisfied by the solution $f_l^{(2)}(r)$ given in equation (2.7b).

The main part of the calculation of the spectrum is the evaluation of Γ, the transmission coefficient. We follow a method analogous to that of Page (1976). The solution to the radial equation for $r \ll a$ is given (we take $2l$ non-integral for convenience (Abramowitz and Stegun 1970))

$$f_l = C_1\, e^{+i\omega r} r^{l+1}\, {}_1F_1(l+1, 2l+2; -2i\omega r) + C_2\, e^{+i\omega r} r^{-l}\, {}_1F_1(-l, -2l; -2i\omega r) \tag{4.2}$$

The constants C_1 and C_2 are evaluated by matching equation (4.1) with the general solution $f_l^{(2)}$ (equation 2.7b) for all modes. We get

$$C_1 = \frac{\Gamma(-\frac{1}{2}-l)}{(2a)^{l+1}\sqrt{\pi}\,\Gamma(-l-\mathrm{i}a\omega)}; \qquad C_2 = \frac{(2U)^l}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+l)}{\Gamma(1+l-\mathrm{i}a\omega)}. \tag{4.3}$$

The asymptotic forms of the confluent hypergeometric functions, ${}_1F_1$, can now be used to get the solution in the form

$$f_l \sim Y_{\text{in}}\,\mathrm{e}^{-\mathrm{i}\omega r} + Y_{\text{out}}\,\mathrm{e}^{\mathrm{i}\omega r}$$

for $\omega r \gg 1$ $(r < a)$

$$Y_{\text{out}} = \exp\left(-\frac{\mathrm{i}}{2}(l+1)\pi\right)\left(\frac{(-a\omega)^{-l-1}}{\Gamma(-l-\mathrm{i}a\omega)} - \frac{\mathrm{i}(a\omega)^l}{\Gamma(1+l-\mathrm{i}a\omega)}\right)$$

$$Y_{\text{in}} = \exp\left(\frac{\mathrm{i}}{2}(l+1)\pi\right)\left(\frac{(-a\omega)^{-l-1}}{\Gamma(-l-\mathrm{i}a\omega)} + \frac{\mathrm{i}(a\omega)^l}{\Gamma(1+l-\mathrm{i}a\omega)}\right).$$

The reflection coefficient R is

$$1-\Gamma \equiv R = \left|\frac{Y_{\text{in}}}{Y_{\text{out}}}\right|^2$$

giving

$$\Gamma = \frac{4A}{(1+A)^2}, \qquad \text{where } A = \frac{(a\omega)^{2l}}{\Pi_1^l[n^2+(a\omega)^2]}. \tag{4.4}$$

The power spectra will be given by equation (4.1).

5. Discussion

The behaviour of the solution $f_l^{(2)}$, used in calculating $\Gamma_l(\omega)$, near the origin $r=0$ is of some interest. We notice from equations (4.2) and (4.3) that

$$f_l^{(2)} \approx C_1 r^{l+1} + C_2 r^{-l} \to C_2 \quad \text{as } r \to 0.$$

The function does not satisfy the requirement that it should vanish as $r \to 0$. This is however not relevant for the following reason.

The field at $r=0$ at any finite time is obtained from the boundary condition on the past horizon which is of the form $\exp(\mathrm{i}\omega U)$ and is hence a function of $v = t + r^*$ only. This leads to the form $\exp(-\mathrm{i}\omega r^*)$ for the spatial part which is seen to correspond to $f_l^{(1)}$ which behaves like r^{l+1} and vanishes as $r \to 0$. Hence the appropriate solution is well behaved at the origin.

We notice from equation (4.4) that for $l=0$, $\Gamma_l(\omega) \sim (a\omega)^{2l}$ and approaches unity. This is in contrast to the black-hole case where $\Gamma_l(\omega) \sim (a\omega)^{2l+1}$. The difference in behaviour is probably due to the difference in effective potential barriers that exist in the case of the de Sitter horizon as compared to the black hole.

Acknowledgment

We are grateful to the referee for very useful suggestions. One of us (DL) is grateful to National Council for Educational Research and Training for financial support.

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